Bessel Processes

Friedrich Bessel (1784-1846)

Let $B_t = (B_t^1, B_t^2, \dots, B_t^d)$ be the standard d-dim Brownian motion, started at some $\vec{x} \neq \vec{0}$.

$X_t := |B_t|$.

By Itô:

$$dX_t^2 = \sum_{j=1}^{d} d\left((B_t^j)^2\right) = 2\sum_{j=1}^{d} B_t^j\, dB_t^j + d\cdot dt = d\cdot dt + 2X_t\, dZ_t,$$

where $Z_t = \sum_{j=1}^{d} \int_0^t \frac{B_s^j}{X_s}\, dB_s^j$.

Then Z_t – local martingale, continuous, $\langle Z, Z\rangle_t = t$.

So Z_t is 1D Brownian motion.

On the other hand, again by Itô,

$$dX_t^2 = 2X_t\, dX_t + d\langle X, X\rangle_t.$$

So
$$dX_t = \frac{dX_t^2 - d\langle X, X\rangle_t}{2X_t} = dZ_t + \frac{d\cdot dt - d\langle X, X\rangle_t}{2X_t}$$

the only term with non-trivial quadratic variation.

So $\langle X, X\rangle_t = t$.

So
$$dX_t = dZ_t + \frac{\left(\frac{d-1}{2}\right)}{X_t}\, dt.$$

$$X_t = |x| + \frac{d-1}{2}\int_0^t \frac{ds}{X_s} + B_t.$$ – Here, it is 1D BM.

General Bessel Process

$$X_t^x = x + B_t + a\int_0^t \frac{ds}{X_s^x}$$ ← take the same BM for all $x > 0$.

Defined for $t \le T_x = \inf\{t : X_t^x = 0\}$.

Existence and uniqueness follows from the following Existence and Uniqueness Theorem on Stochastic Differential Equations:

Thm Let (B_t) be standard 1D BM, $F, G : [0,T] \to \mathbb{R}$ -

measurable. Assume that

1) $|F(t,x)| + |G(t,x)| \lesssim (1+|x|)$

2) F, G are Lipshitz in x, uniformly in t.

Let Z be independent of (B_t), $E(Z^2)<\infty$.

Then the SDE

$$dX_t = F(t,X_t)\,dt + G(t,X_t)\,dB_t$$

has unique solution with $X_0 = Z$.

(X_t) is adapted with respect to $\left(\mathcal{F}_t^Z = \sigma(\mathcal{F}_t, Z)\right)_{t\geq 0}$

$E\left(\int_0^T X_t^2\,dt < \infty\right)$.

<u>For Bessel case:</u> $G \equiv 1$, $F(t,X) = \frac{a}{X}$ – not Lipshitz,

so consider $F^n(t,x) = \min\left(\left|\frac{a}{x}\right|, n\right)$ – Lipshitz, solution agrees with Bessel up to $T_n = \inf\{t : |X_t| < \left|\frac{a}{n}\right|\}$.

<u>Assume</u> : $X > 0$

<u>Observe</u>: 1) $x < y \Rightarrow X_t^x < X_t^y$ for all $t < T_x$

(For $t < T_x$,

$$X_t^y - X_t^x = y - x - \int_0^t \frac{a\,(X_s^y - X_s^x)}{X_s^y X_s^x}\,ds$$

So $X_t^y - X_t^x$ is differentiable,

$$\frac{d}{dt}(X_t^y - X_t^x) = -(X_t^y - X_t^x)\frac{a}{X_t^x X_t^y}, \text{ so}$$

$$X_t^y - X_t^x = (y-x)\exp\left(-a\int_0^t \frac{ds}{X_s^x X_s^y}\right) > 0).$$

So, in particular, $T_x \leq T_y$. But could be $T_x = T_y$!

2) Brownian scaling:

$X_t^1 = x^{-1} X_{x^2 t}^x$ (just plug in the equation!)
in law

<u>Theorem</u>

1) If $a \geq \frac{1}{2}$, then $P(T_x = \infty \text{ for all } x) = 1$.

2) $a \leq \frac{1}{2} \Rightarrow \inf_t X_t^x = 0$ a.s

3) $a > \frac{1}{2} \Rightarrow X_t^x \xrightarrow[t\to\infty]{} \infty$

4) $a < \frac{1}{2} \Rightarrow P(T_x < \infty \text{ for all } x) = 1$

5) $\frac{1}{4} < a < \frac{1}{2}$, $x < y \implies P(T_x = T_y) > 0$

6) $a \le \frac{1}{4}$ $\quad P(\forall x < y;\ T_x < T_y) = 1.$

Proof. Fix $0 < x_1 < x < x_2 < \infty$.

Let $\tau = \inf\{t : X_t^x \in \{x_1, x_2\}\}$ - first time X_t^x hits x_1 or x_2.

$\varphi(x) := \varphi(x, x_1, x_2) = P(X_\tau^x = x_2)$

$\varphi(X_{t\wedge\tau}^x) = E(\varphi(X_\tau^x) | \mathcal{F}_t)$ - martingale.

If $\varphi \in C^2$, then by Itô:

$d\,\varphi(X_{t\wedge\tau}^x) = \varphi'(X_{t\wedge\tau}^x)\,dX_{t\wedge\tau}^x + \frac{1}{2}\varphi''(X_{t\wedge\tau}^x)\,dt =$

$\underbrace{\varphi'(X_{t\wedge\tau}^x)\,dB_t}_{\text{martingale}} + \underbrace{\left(\frac{a\,\varphi'(X_{t\wedge\tau})}{X_t^x} + \left(\frac{\varphi''(X_t)}{2}\right)\right)dt}_{\text{Bounded Variation.}}$

So, since φ is a martingale,

$\boxed{\frac{1}{2}\varphi''(x) + \frac{a}{x}\varphi'(x) = 0, \quad x_1 < x < x_2, \quad \varphi(x_1) = 0, \quad \varphi(x_2) = 1.}$ $(*)$

We don't know apriori that $\varphi \in C^2$!

Trick: solve $(*)$

$$\varphi_0(x) = \begin{cases} \dfrac{x^{1-2a} - x_1^{1-2a}}{x_2^{1-2a} - x_1^{1-2a}}, & a \ne \frac{1}{2} \\ \dfrac{\log x - \log x_1}{\log x_2 - \log x_1}, & a = \frac{1}{2}. \end{cases}$$

By Itô: $M_t := \varphi_0(X_{t\wedge\tau}^x)$ - bounded martingale (drift term disappears!).

So $P(X_\tau^x = x_2) = E(M_\infty | \mathcal{F}_0) = \varphi_0(x)$ $\quad M_\infty = \varphi_0(X_\tau^x) = \begin{cases} 1, & X_\tau^x = x_2 \\ 0, & X_\tau^x = x_1 \end{cases}$

So $\varphi(x, x_1, x_2) = \varphi_0(x)$.

Observe now:

$\lim_{x_1 \to 0} \varphi_0(x, x_1, x_2) = \begin{cases} 1, & a \ge \frac{1}{2} \text{ for any } x_2, \text{ so } T_x = \infty \\ \left(\frac{x}{x_2}\right)^{1-2a}, & a < \frac{1}{2} \quad \text{so } T_x < \infty \text{ a.s} \end{cases}$ | 1) 4) > proven for fixed x, so a.s for all rational x, use monotonicity: $x<y \Rightarrow T_x \le T_y$.

$\lim_{x_2 \to \infty} \varphi_0(x, x_1, x_2) = \begin{cases} 0, & a \le \frac{1}{2}, \text{ so } \inf X_t^x = 0 \Rightarrow 2) \\ 1 - \left(\frac{x_1}{x}\right)^{2a-1}, & a > \frac{1}{2} \Rightarrow \lim_{t\to\infty} X_t^x = \infty \text{ a.s.} \end{cases}$

Consider $a > \frac{1}{2}$. Let us prove 3): $\lim_{t\to\infty} X_t^x = \infty$.

Let $T_n := \inf\{t : X_t^x = 2^n\}$.

Consider $a > \frac{1}{2}$. Let us prove 3): $\lim_{t\to\infty} X_t^x = \infty$.

Let $T_n := \inf\{t : X_t^x = 2^n\}$.

$\ell(x_1,y) := \lim_{x_2\to\infty} \varphi_0(y, x_1, x_2) = 1 - \left(\frac{x_1}{y}\right)^{2a-1}$

$|\varphi_0(y, x_1, x_2) - \ell(x_1,y)| \le \left(y^{1-2a} - x_2^{1-2a}\right)\left(\frac{x_1}{x_2}\right)^{2a-1}$

so $P(\exists t : T_n \le t \le T_{n+1},\ X_t^x < \frac{2^n}{M_n}) = \varphi(2^n, \frac{2^n}{M_n}, 2^{n+1}) < \frac{C}{M_n^{2a-1}}$

Take $M_n := n^{\frac{2}{2a-1}}$ to get

$\sum_n P(\exists t : T_n \le t \le T_{n+1},\ X_t^x < \frac{2^n}{n^{2/(2a-1)}}) < \infty$. So, by

Borel-Cantelli, a.s. $\exists N$: $\forall n > N$ $\inf_{T_n \le t \le T_{n+1}} X_t^x \ge \frac{2^n}{n^{2/(2a-1)}} \to \infty$,

so $\lim_{t\to\infty} X_t^x = \infty$.

Now look at $a < \frac{1}{2}$.

Let $q(x,y) := P(T^x = T^y)$

By scaling, $q(x,y) = q(\frac{x}{y}, 1)$.

Note that $\lim_{r\to 0} q(r,1) = 0$.

Indeed, T^r has the same distribution as $r^2 T^1$ (by scaling).

So $\forall \varepsilon > 0$ $\exists r, \delta$: $P(T^r \ge \delta) \le \frac{\varepsilon}{2}$, $P(T^1 \le \delta) \le \frac{\varepsilon}{2}$, so $P(T_1 = T^r) \le \varepsilon$

Let now $u := \sup_{t < T_r} \frac{X_t^1}{X_t^r}$.

Claim $P(T_r < T_1,\ u < \infty) = 0$ i.e. $(T_r = T_1) \overset{a.s.}{\Leftarrow} (u < \infty)$.

$P(T_r = T_1,\ u = \infty) = 0$

Proof. If $u < \infty$ then $X_t^1 \le c X_t^r$, for all t, so $T^1 = T^r$.
This gives the first identity.

For the second, let $\sigma_N := \inf\{t : \frac{X_t^1}{X_t^r} = N\}$.

Then $P(u \ge N : T_1 = T_r) \le P(T^1 = T^r \mid \sigma_N < \infty) \overset{\text{Markov + rescaling}}{=}$

$q(\frac{1}{N}, 1) \to 0$ as $N \to \infty$

Let $L_t := \log\left(\frac{X_t^1}{X_t^r} - 1\right) = \log(X_t^1 - X_t^r) - \log X_t^r$.

By the definition of Bessel and Itô formula

$$dL_t = \frac{1}{(X_t^r)^2}\left(\frac{1}{2} - a - \frac{a}{e^{L_t}+1}\right)dt - \frac{1}{X_t^r}\,dB_t.$$

Change time to make martingale term $\left(\frac{dB_t}{X_t^r}\right)$ a standard BM:

define $\sigma(t)$ as $\int_0^{\sigma(t)} \frac{ds}{(X_s^r)^2} = t$.

Observe that $\sigma^{-1}(T^r) = \infty \iff \int_0^{T^r} \frac{ds}{(X_s^r)^2} = \infty$

Indeed, let $T_j := \inf\{t : X_t^r = 2^{-j}\}$, $Y_j := \int_{T_{j-1}}^{T_j} \frac{ds}{(X_s^r)^2} = \int_0^{T_j - T_{j-1}} \frac{ds}{(X_s^{2^{-j}})^2}$.

By scaling, Y_j are i.i.d, $E(Y_j) > 0$, so $\sum Y_j = \int_0^{T^r} \frac{ds}{(X_s^r)^2} = \infty$.

$$W(t) = -\int_0^{\sigma(t)} \frac{dB_s}{X_s^r}.$$

Then $W(t)$ – BM, $\tilde{L}_t := L_{\sigma(t)}$.

$$d\tilde{L}_t = \left(\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1}\right) dt + dW_t.$$

$\forall a < \frac{1}{2}\ \exists \alpha > 0, K < \infty : \tilde{L}_t \geq K \Rightarrow \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} > \alpha$

So if $\tilde{L}_t \geq K+1$, then, comparing with B.M. with drift α $(B_t + \alpha t)$ we see that $\tilde{L}_t \to \infty$ with positive probability ($B_t + \alpha t$ hits ∞ before K with positive probability). But $\tilde{L}_t \to \infty \iff \frac{X_t^1}{X_t^r} \xrightarrow[t \to T^r]{} \infty$.

Moreover, for $a \leq \frac{1}{4}$,

$\frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} > 0$, so $\tilde{L}_t$ dominates ~~t~~he usual B.M,

and so $\overline{\lim}\, \tilde{L}_t = \infty$

But if $\tilde{L}_t \geq K+1$ then $\lim_{t \to \infty} \tilde{L}_t = \infty$ with positive probability.

So, since $\tilde{L}_t \geq K+1$ infinitely often, $\lim_{t \to \infty} \tilde{L}_t = \infty$ with probability one, so $= \infty$ a.s, which implies 6)

For $a > \frac{1}{4}$, then $\exists K > 0, \alpha > 0$, such that $\tilde{L}_t \leq -K \Rightarrow \frac{1}{2} - a - \frac{a}{e^{\tilde{L}_t} + 1} < -\alpha$.

By again comparing with $B_t - \alpha t$, we see that $\tilde{L}_t \leq -K-1 \Rightarrow$ with positive probability $\tilde{L}_t \to -\infty. \Rightarrow u < \infty \underset{a.s.}{\Rightarrow} T_r = T_1$.

But for any ℓ, $P(\tilde{L}_t \leq -K-1 \mid L_0 = \ell) > 0$ (by comparing now with $B_t + \frac{1}{4} t$).

So in any case $P(u<\infty)>0$.

This means : $\forall r$: $\varphi(r,1)>0$.

By monotonicity and scaling, it leads to 5.1 $\blacksquare$